

Products of Quasinormal Groups (QSG)

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ABSTRACT

A subgroup H of a group G is termed quasinormal in G if it satisfies the following equivalent conditions:

1. For any subgroup K of G , HK (the product of subgroups H and K) is a group
2. For any subgroup K of G , $HK=KH$, i.e., H and K are permuting subgroups.
3. For every $g \in G$, H permutes with the cyclic subgroup generated by g . In symbols, for every $h \in H$ and $g \in G$, there exists $h' \in H$ and an integer n such that $hg = g^n h'$.

We say that $G=AB$ is the mutually permutable product of the subgroups A and B if A permutes with every subgroup of B and B permutes with every subgroup of A . We say that the product is totally permutable if every subgroup of A permutes with every subgroup of B . In this paper we prove the following theorem.

Let $G=AB$ be the mutually permutable product of the super soluble subgroups A and B . If $\text{Core } G(A \cap B)=1$, then G is super soluble.

Introduction

All groups considered in this paper are finite. It is known that a group which is the product of two super soluble groups is not necessarily super soluble, even if the two factors are normal subgroups of the group. Baer proved in that if a group G is the product of two normal supersoluble groups and G' is nilpotent, then G is super soluble. The search for generalisations of Baer's result has been a fruitful topic of investigation recently [1-3].

Most of the generalisations centre around replacing normality of the factors by different permutability conditions. In Asaad and Shaalan considered products satisfying one of the following conditions [4]. We will follow Carocca, and say that $G=AB$ is the mutually permutable product of the subgroups A and B if A permutes with every subgroup of B and B permutes with every subgroup of A [5]. We say that the product is totally permutable if every subgroup of A permutes with every subgroup of B . Essentially, the results by Asaad and Shaalan are devoted to obtaining sufficient conditions for a mutually permutable product of two supersoluble subgroups to be supersoluble. They prove in [2, Theorem 3.8] the following generalisation of Baer's theorem.

Let G be the mutually permutable product of the supersoluble subgroups A and B . If G' is nilpotent, then G is supersoluble. They also show that the result remains true if " G' nilpotent" is replaced by " B nilpotent" [2, Theorem 3.2]. In addition, they prove [2, Theorem 3.1]: If G is the totally permutable product

of the supersoluble subgroups A and B , then G is supersoluble. It is well known that if $G=AB$ is a mutually permutable product of two supersoluble subgroups A and B such that $A \cap B=1$, then the product is in fact totally permutable [6, Proposition 3.5], and therefore G is supersoluble. Our main Theorem is a generalisation of this last property.

Theorem 1

Let $G=AB$ be the mutually permutable product of the supersoluble subgroups A and B . If $\text{Core } G(A \cap B)=1$, then G is supersoluble.

The second aim of the present paper has been to obtain more complete information about the structure of mutually permutable products of two supersoluble groups.

As a straight forward consequence of Theorem 1, we have that, in the notation used above, $G/\text{Core } G(A \cap B)$ is always supersoluble. Therefore every mutually permutable product of two supersoluble subgroups is metasupersoluble. It is possible to obtain more precise information about its structure, as our second main theorem shows.

Theorem 2. Let $G=AB$ be the mutually permutable product of the supersoluble subgroups A and B . Then $G/F(G)$ is supersoluble and metabelian. This last theorem can not be improved easily, as the following example shows.

Example. Let S_3 be the symmetric group of degree 3, given by $S_3 = \langle \alpha, \beta : \alpha^2 = \beta^3 = 1; \beta\alpha = \beta^2 \rangle$ and let T_7 be the non-abelian group of order 73 and exponent 7. Write $T_7 = \langle a, b \rangle$ with $a^7 = b^7 = [a, b]^7 = 1$ and let $c = [a, b]$. We have that S_3 acts on T_7 in the following way: $\alpha a = b, \beta a = a, \alpha b = c^{-1}, \alpha^2 b = a, \beta b = b^4, c\beta = c$.

Thus we can consider the semidirect product $G = [T_7] S_3$. Take now the subgroups $A = T_7 \langle \beta \rangle$ and $B = T_7 \langle \alpha \rangle$ of G . Clearly both A and B are supersoluble, and it is easy to check that $G = AB$ is the mutually permutable product of A and B . Finally, we show that Theorem 1 provides elementary proofs for the results of Asaad and Shaalan about mutually permutable products.2.

Main results The following four lemmas are needed to prove Theorem 1. **Lemma 1** [4, Theorem 2]. If $G = AB$ is the mutually permutable product of the supersoluble subgroups A and B , then G is soluble.

Lemma 2. Let $G = AB$ be the mutually permutable product of the supersoluble subgroups A and B . Then, either G is supersoluble or $NA < G$ and $NB < G$ for every minimal normal subgroup N of G .

Proof. Assume that G is not supersoluble. Then both A and B are proper subgroups of G . Let N be a minimal normal subgroup of G and for contradiction assume that $NA = G$. Then, as N is abelian, $N \cap A$ is a normal subgroup of $\langle N, A \rangle = G$. Since N is a minimal normal subgroup of G and $A < G$, we have that $N \cap A = 1$ and consequently A is a maximal subgroup of G . Clearly we can also assume that B is not contained in A . It is not difficult to argue that we can choose an element b of $B \setminus A$ such that $bq \in A$ for some prime q . Since the product $G = AB$ is mutually permutable, $A \langle b \rangle$ is a subgroup of G and the maximality of A implies that $G = A \langle b \rangle$. We now take orders to reach our final contradiction: $|A||N| = |G| = |A| \langle b \rangle |A \cap \langle b \rangle| = q|A|$. Consequently we have that $|N| = q$ and then G is supersoluble, a contradiction.

Lemma 3. Let $G = AB$ be the mutually permutable product of the subgroups A and B and let N be any minimal normal subgroup of G . Then either $N \cap A = N \cap B = 1$ or $N = (N \cap A)(N \cap B)$.

Proof. Let N be a minimal normal subgroup of G . Clearly $A(N \cap B)$ and $(N \cap A)B$ are both subgroups of G . Note that A normalizes $N \cap (A(N \cap B)) = (N \cap A)(N \cap B)$ and B normalizes $N \cap ((A \cap N)B) = (N \cap A)(N \cap B)$. Therefore $(N \cap A)(N \cap B)$ is a normal subgroup of G and the minimality of N yields the result.

Lemma 4. Let G be a group, and N a minimal normal subgroup of G such that $|N| = p^n$, where p is a prime and $n > 1$. Denote $C = CG(N)$ and assume that G/C is supersoluble. Then, if Q/C is a subgroup of G/C containing $Op'(G/C)$, we have that Q is normal in G and $N = \prod_{i=1}^t N_i$, where N_i are non-cyclic minimal normal subgroups of NQ for $i = 1, \dots, t$.

Proof. Since by [8, Lemma A.13.6], we have that $Op(G/C) = 1$ and the commutator subgroup $(G/C)'$ of G/C is nilpotent because $e/G/C$ is supersoluble, it follows that $(G/C)'$ is a p' -group. Therefore $(G/C)'$ is contained in $Op'(G/C)$ and thus $Op'(G/C)$

is a Hall p' -subgroup of G/C . Consequently, Q/C is a normal subgroup of G/C and hence Q is normal in G . Consider now N as a G -module over $GF(p)$ by conjugation. Then, by Clifford's Theorem [8, Theorem B.7.3], N viewed as a Q -module is a direct sum $N = \prod_{i=1}^t N_i$, where N_i are irreducible Q -modules for $i = 1, \dots, t$. Suppose that there exists $i \in \{1, \dots, t\}$ such that $|N_i| = p$. Then clearly $|N_j| = p$ for all j . Therefore $Q/CQ(N_i)$ is abelian of exponent dividing p^{-1} , and the same is true for Q/C . In particular, $Q/C = Op'(G/C)$ is a Hall p' -subgroup of G/C . Since N is not cyclic, it follows that $Q = G$ and thus p divides $|G/C|$. Hence p is the largest prime dividing $|G/C|$. From the supersolubility of G/C , we obtain that $1 = Op(G/C)$ is a Sylow subgroup of G/C , a contradiction. Consequently, N_i is a non-cyclic minimal normal subgroup of NQ for all $i \in \{1, \dots, t\}$, as we wanted to prove.

Proof of Theorem 1. Let $G = AB$ be the mutually permutable product of the supersoluble subgroups A and B , with $CoreG(A \cap B) = 1$, and suppose that G has been chosen minimal such that its supersoluble residual GU is non-trivial. Let N be a minimal normal subgroup of G contained in GU . Note that N is an elementary abelian p -group for some prime p . Applying Lemma 2, we have that both NA and NB are proper subgroups of G . Moreover, using Lemma 3, we have that either $N = (N \cap A)(N \cap B)$ or $N \cap A = N \cap B = 1$. Assume first that $N = (N \cap A)(N \cap B)$.

1. If $N \cap A = 1$, then N is cyclic. Assume that $N \cap A = 1$. It follows that N is contained in B . Let N_0 be a non-trivial cyclic subgroup of N . Since AN_0 is a subgroup of G , we have that $N_0 = AN_0 \cap N$ is a normal subgroup of AN_0 . Hence every cyclic subgroup of N is normalised by A . Now let N_1 be a minimal normal subgroup of B contained in N . Since B is supersoluble, it follows that N_1 is cyclic and thus normalised by A . Hence N_1 is a normal subgroup of G . The minimality of N implies that $N = N_1$ and consequently N is cyclic.
2. $N \cap A = 1$ and $N \cap B = 1$. On the contrary, assume that $N \cap A = 1$. From (i), we know that N is cyclic. Moreover, N is contained in B . Hence $AN \cap B = (A \cap B)N$. Let $L = CoreG(A \cap B)N$. Clearly, N is contained in L and $L = L \cap ((A \cap B)N) = (L \cap A \cap B)N$. It is clear that $G/L = (AL/L)(BL/L)$ is a mutually permutable product of AL/L and BL/L such that $CoreG/L((AL/L) \cap (BL/L)) = 1$. By the minimality of G , it follows that G/L is supersoluble. On the other hand, since N is cyclic, we have that $G/CG(N)$ is abelian. Hence $G/CL(N)$ is supersoluble and $GUCL(N) = C$. Note that $C = N \times (C \cap A \cap B)$. Therefore $C \cap A \cap B$ contains a Hall p' -subgroup of C . Since $CoreG(A \cap B) = 1$ and $Op'(C)$ is a normal subgroup of G contained in $C \cap A \cap B$, we have that $Op'(C) = 1$. Moreover, $C' = (C \cap A \cap B)'$ is a normal subgroup of G contained in $A \cap B$. Consequently, $C' = 1$ and C is an abelian p -group. In particular, GU is abelian and thus GU is complemented in G by a supersoluble normalizer D which is also a supersoluble projector of G , by [8, Theorems V.4.2 and V.5.18]. Since N is cyclic, we know that N is central with respect to the saturated formation of all supersoluble groups. By [8, Theorem V.3.2.e], D covers N and thus N is contained in D . It follows $ND \cap GU = 1$, a contradiction.
3. Either $N = N \cap A$ or $N = N \cap B$. If we have $N = N \cap A = N \cap B$, then N is contained in $A \cap B$, contradicting the fact that $CoreG(A \cap B) = 1$. We may assume without loss of generality that $N \cap A = N$.

4. AN and BN are both supersoluble. Since $N=(N \cap A)(N \cap B)$ and $N=N \cap A$, it follows that $N \cap B$ is not contained in $N \cap A$. Let n be any element of $N \cap B$ such that $n \notin N \cap A$, and write $N_0 = \langle n \rangle$. Note that AN_0 is a subgroup of G , and $AN_0 \cap N = (N \cap A)N_0$. Therefore $N_0(N \cap A)$ is a normal subgroup of AN_0 , and consequently A normalizes $(A \cap N)N_0$. This yields that $A/CA(N/N \cap A)$ acts as a power automorphism group on $N/N \cap A$. This means that AN is supersoluble. If $N \cap B=N$, then $BN=B$ is supersoluble. On the contrary, if $N \cap B \neq N$, we can argue as above and we obtain that BN is supersoluble. Consequently, $ACG(N)/CG(N)$ and $BCG(N)/CG(N)$ are both abelian groups of exponent dividing p^{-1} . But then $G/CG(N) = (ACG(N)/CG(N))(BCG(N)/CG(N))$ is a π -group for some set of primes π such that if $q \in \pi$, then q divides p^{-1} .
5. Let B_0 be a Hall π -subgroup of B . Then $AB_0 \cap N = A \cap N$. This follows just by observing that $AB_0 \cap N$ is contained in each Hall π' -subgroup of AB_0 and every Hall π' -subgroup of A is a Hall π' -subgroup of AB_0 . Note that $|G/CG(N)|$ is a π -number and AB_0 contains a Hall π -subgroup of G . Therefore $G = (AB_0)CG(N)$. But then $A \cap N$ is a normal subgroup of G . The minimality of G yields either $A \cap N = 1$ or $A \cap N = N$. This contradicts our assumption $1 = N \cap A = N$, and so we cannot have $N = (A \cap N)(B \cap N)$. Thus, by Lemma 3, we may assume $N \cap A = N \cap B = 1$. Let $M = \text{Core}G(A \cap B \cap N)$. Then $N \cap M = N$ and G/M is supersoluble by the minimality of G . Again, we reach a contradiction after several steps.
6. $M=N$. Suppose that $M \neq N$. Since G/M is supersoluble, we know that N cannot be cyclic. Let us write $C = CG(N)$, and consider the quotient group G/C . It is clear that G/C is supersoluble. Let $Q/C = \text{Op}(G/C)$. Since $\text{Op}(G/C) = 1$ and (G/C) is nilpotent, it follows that Q/C is a normal Hall p' -subgroup of G/C . Let $B_{p'}$ be a Hall p' -subgroup of B . Since $|N|$ divides $|B: A \cap B|$, we have that $(A \cap B)B_{p'}$ is a proper subgroup of B . Let T be a maximal subgroup of B containing $(A \cap B)B_{p'}$. Then AT is a maximal subgroup of G and $|G: AT| = p = |B: T|$. If N is not contained in AT , we have $G = (AT)N$ and $AT \cap N = 1$. Then $|N| = p$, a contradiction. Therefore N is contained in AT . In particular, the family $S = \{X: X \text{ is a proper subgroup of } B, (A \cap B)B_{p'} \leq X \text{ and } NAX\}$ is non-empty. Let R be an element of S of minimal order. Observe that AR has p -power index in G and thus ARC/C contains $\text{Op}'(G/C)$. Regarding N as a AR -module over $GF(p)$, we know, by Lemma 4, that N is a direct sum $N = \prod_{i=1}^t N_i$, where N_i is an irreducible AR -module whose dimension is greater than 1, for all $i \in \{1, \dots, t\}$. Assume that $(A \cap B)B_{p'} = R$. Then $AR = AB_{p'}$ and thus N is contained in A , a contradiction. Therefore $AB_{p'} \cap B = (A \cap B)B_{p'}$ is a proper subgroup of R . Let S be a maximal subgroup of R containing $(A \cap B)B_{p'}$. From the minimality of R , we know that N is not contained in AS . Consequently, there exists some $i \in \{1, \dots, t\}$ such that N_i is not contained in AS , which is a maximal subgroup of AR . Hence $AR = (AS)N_i$. Since N_i is a minimal normal subgroup of AR , it follows that $AS \cap N_i = 1$ and $|N_i| = |AR: AS| = |R: S| = p$, a contradiction.
7. M is an elementary abelian p -group. Note that $M = N(M \cap A) = N(M \cap B)$ and $|M \cap A| = |M \cap B| = |M|/|N|$. Moreover, $A(M \cap B)$ is a subgroup of G such that $A(M \cap B) \cap M = (M \cap A)(M \cap B)$. Hence $(M \cap A)(M \cap B)$ is also a subgroup of G . If $M \cap A = M \cap B$, then $M \cap A$ is a

normal subgroup of G contained in $A \cap B$. This implies that $M \cap A = 1$ and consequently $M = N$, a contradiction. It yields that $M \cap A \neq M \cap B$. Next we see that $(M \cap A)(M \cap B)$ is a normal subgroup of G . Since $(M \cap A)(M \cap B) = M \cap A(M \cap B)$, we have that A normalizes $(M \cap A)(M \cap B)$.

Similarly, B normalizes $(M \cap A)(M \cap B)$ since $(M \cap A)(M \cap B) = M \cap B(M \cap A)$. This implies normality of $(M \cap A)(M \cap B)$ in G . Let $X = (M \cap A)(M \cap B)$. Since we cannot have $M \cap A = M \cap B$, $M \cap A$ must be strictly contained in X . Thus $X = X \cap M = (X \cap N)(M \cap A) > M \cap A$ gives us $X \cap N = 1$. But then $X \cap N = N$, giving NX . Suppose that Q is a Hall p' -subgroup of $M \cap B$. Then QA is a subgroup and so $QA \cap M = Q(M \cap A)$ is also a subgroup which contains Q . Hence, as $|M: M \cap A| = p^k$ for some k , we have that $QM \cap A \cap B$. Thus $QB \cap MM \cap A \cap B$ and similarly $QA \cap MM \cap A \cap B$. Consequently, QM is contained in $M \cap A \cap B$. Since $QM = \text{Op}(M)$, it follows that $\text{Op}(M)$ is a normal subgroup of G contained in $A \cap B$. Hence $\text{Op}(M) = 1$, a contradiction, and consequently $Q = 1$ and M is a p -group. Hence N is contained in $Z(M)$ and $M = N \times (M \cap A) = N \times (M \cap B)$. Thus $\phi(M) = \phi(M \cap A) = \phi(M \cap B)$ is a normal subgroup of G contained in $A \cap B$. This implies that $\phi(M) = 1$ and M is an elementary abelian p -group, as claimed.

8. Final contradiction. We have from the previous steps that $M \cap A$ is not contained in $M \cap B$ and that $M \cap B$ is not contained in $M \cap A$ because otherwise, since $|M \cap A| = |M \cap B|$, it follows that $M \cap A = M \cap B$ is a normal subgroup of G contained in $A \cap B$. This would imply $M \cap A = M \cap B = 1$, and $M = (M \cap A)N = N$. This fact contradicts step (vi).

Let x be an element of $M \cap B$ such that $x \notin M \cap A$. Then $A \langle x \rangle$ is a subgroup of G , and so is $M_0 = A \langle x \rangle \cap M = (A \cap M) \langle x \rangle$. Therefore M_0 is an A -invariant subgroup of G . In particular, since $M = (M \cap A)(M \cap B)$, we have that every subgroup of $M/M \cap A$ is A -invariant; that is, $A/CA(M/M \cap A)$ acts as a group of power automorphisms on $M/M \cap A$. It is clear that $M/M \cap A$ is A -isomorphic to N . Consequently, $A/CA(N)$ acts as a group of power automorphisms on N . This implies that A normalises each subgroup of N . Analogously, B normalises each subgroup of N . It follows that N is a cyclic group. We argue as in step (ii) above to reach a final contradiction. We have that G/M is supersoluble and M is abelian.

Therefore GUM and thus GU is abelian and complemented in G by a supersoluble normaliser, D say, by [8, Theorem V.5.18]. Since N is cyclic, we know that D covers N and thus $NGU \cap D = 1$, a contradiction. Proof of Theorem 2.

Let $M = GU$ denote the supersoluble residual of G . Theorem 1 yields that $G/\text{Core}G(A \cap B)$ is supersoluble. Therefore M is contained in $\text{Core}G(A \cap B)$. In particular, M is supersoluble. Let $F(M)$ be the Fitting subgroup of M . Since A and B are supersoluble, we have that $[M, A]F(A) \cap MF(M)$ and $[M, B]F(B) \cap MF(M)$.

Consequently, $[M, G]$ is contained in $F(M)$. Note now that the chief factors of G between $F(M)$ and M are cyclic, and recall that G/M is supersoluble. Therefore we have that $G/F(M)$ is supersoluble. This implies that $M = F(M)$ and thus M is nilpotent.

Consequently, G/F (G) is supersoluble. We now show that G/F (G) is metabelian. We prove first that A' and B' both centralise every chief factor of G . Let H/K be a chief factor of G . If H/K is cyclic, then as G' centralizes H/K , so do A' and B' . Hence we may assume that H/K is a non-cyclic p -chief factor of G for some prime p . Note that we may assume that H is contained in M because G/M is supersoluble and H/K is non-cyclic. To simplify notation, we can consider $K=1$. Since $F(G)$ centralizes H [8, Theorem A.13.8.b], $G/CG(H)$ is supersoluble. Let Ap' be a Hall p' -subgroup of A . By Maschke's theorem [8, Theorem A.11.5], H is a completely reducible Ap' -module and HAp' is supersoluble because H is contained in A . Therefore $Ap'/CAp'(H)$ is abelian of exponent dividing $p-1$. This implies that the primes involved in $|A/CA(H)|$ can only be p or divisors of $p-1$. The same is true for $|B/CB(H)|$. This implies that if p divides $|G/CG(H)|$, then p is the largest prime dividing $|G/CG(H)|$. But since $Op(G/CG(H))=1$ and $G/CG(H)$ is supersoluble, it follows that $G/CG(H)$ must be a p' -group. Consider H as A -module over $GF(p)$. Since $ACG(H)/CG(H)$ is a p' -group, we have that H is a completely reducible A -module and every irreducible A -submodule of H is cyclic. Consequently A' centralizes H , and the same is true for B' . Let now U/V be a chief factor of G . Then $G/CG(U/V)$ is the product of the abelian subgroups $ACG(U/V)/CG(U/V)$ and $BCG(U/V)/CG(U/V)$. By Itô's theorem [9], we have that $G/CG(U/V)$ is metabelian. Since $F(G)$ is the intersection of the centralisers of all chief factors (again by [8, Theorem A.13.8.b]), we can conclude that G/F (G) is metabelian. 3. Final remarks Finally, Theorem 1 enables us to give succinct proofs of earlier results on mutually permutable products.

Corollary 1 [2, Theorem 3.2]. Let $G=AB$ be the mutually permutable product of the subgroups A and B . If A is supersoluble and B is nilpotent, then G is supersoluble.

Proof. Assume that the assertion is false, and let G be a minimal counterexample. We have that G is a primitive group, and so G has a unique minimal normal subgroup, N say, with $N=CG(N)$ a p -group for some prime p . Since G is not supersoluble, applying Theorem 1, we know that $Core_G(A \cap B)=1$. This yields that N is contained in $A \cap B$. Now, since N is contained in B , which is nilpotent, it follows that any p' -element of B must centralize N . Since $CG(N)=N$, we have that B itself is a p -group. Consequently, A must contain a Hall p' -subgroup of G . Now let $T/N=Op'(G/N)$. The previous argument yields that T/N is contained in A/N . Note that if $B=N$, then $G=AN=A$ is supersoluble, a contradiction. Thus N is a proper subgroup of B . This implies that p must divide $|G:T|$. Since G/N is supersoluble, p must divide $q-1$ for some prime $q \in \pi(T/N)$. It is clear then that q can not divide $p-1$. Therefore there exists a Sylow q -subgroup A_q of A which centralizes N . Using that $CG(N)=N$, it yields that $A_q=1$ and thus q does not divide $|G|$, a contradiction.

Corollary 2 [2, Theorem 3.8]. Let $G=AB$ be the mutually permutable product of the supersoluble subgroups A and B . If G' is nilpotent, then G is supersoluble.

Proof. We assume the result to be false, and choose a minimal counterexample G . Thus G is a primitive group with unique minimal normal subgroup N . We also have that $G=NM$, where M is a maximal subgroup of G , $N \cap M=1$ and $N=F(G)=Op(G)$ for some prime p . Now G' is nilpotent and thus $G'=F(G)=N$. Therefore M is an abelian group. Since N is self-centralising, arguing as we did in the previous corollary, we have that N is contained in $A \cap B$. Note that $M \in G/N$, and thus $Op(M)=1$. Since M is abelian, this yields that M is a p' -group. Thus M is in fact a Hall p' -subgroup of G . Applying [1, Theorem 1.3.2], we have that there exist a Hall p' -subgroup Ap' of A and a Hall p' -subgroup Bp' of B such that $M=Ap'Bp'$. Since $NA \cap B$, it follows that both Ap' and Bp' must have exponent dividing $p-1$. Regarding N as a M -module, it is easy to see that M must be a cyclic group. Now, since $M=Ap'Bp'$ has exponent dividing $p-1$, it follows that N is a cyclic group as well. This implies that G is supersoluble, a contradiction.

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